# Zeno Paradox for Bohmian Trajectories: The Unfolding of the Metatron

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**Abstract**

We study an analogue of the quantum Zeno paradox for the Bohm trajectory of a sharply located particle. We show that a continuously observed Bohm trajectory is the classical trajectory. Our results are then compared and contrasted with Mott's original treatment of an $\alpha$-particle track produced by a decaying nucleus in a cloud chamber using standard quantum mechanics.

## 1 Introduction

Einstein writes to Bohm in 1954,

> *I am glad that you are deeply immersed seeking an objective description of the phenomena and that you feel the task is much more difficult as you felt hitherto. You should not be depressed by the enormity of the problem. If God had created the world his primary worry was certainly not to make its understanding easy for us. I feel it strongly since fifty years.*[13]

When David Bohm completed his book, "Quantum Theory" [4], which was an attempt to present a clear account of Bohr's actual position, he became dissatisfied with the overall approach [6]. The reason for this dissatisfaction was the fact that the theory had no place in it for an adequate

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notion of an independent actuality, that is of an actual movement or activity by which one physical state could pass over into another.

In a meeting with Einstein, ostensibly to discuss the content of his book, the conversation eventually turned to the possibility of whether a deterministic extension of quantum mechanics could be found. Later while exploring the WKB approximation, Bohm realised that this approximation was giving an essentially deterministic approach. Surely by merely truncating a series, one cannot turn a probabilistic theory into a deterministic theory. Thus by retaining all the terms in the series, Bohm found that one could, indeed, obtain what looked like a deterministic description of quantum phenomena. To carry this through, he had to assume that a quantum particle actually *had* a well defined but unknown position and momentum and followed a well-defined trajectory.

In the simple approach to the Bohm model, the Schrödinger equation is cast into a form that brings out its close relationship to the classical Hamilton-Jacobi theory, the only difference being an additional term which can be regarded as a new quality of energy, called the 'quantum potential energy'. It is the properties of this energy that enables us to account for all quantum phenomena such as, for example, the two-slit interference effect where the trajectories are shown to undergo a non-classical behaviour [41].

In this paper we will show that if we continuously observe a Bohm trajectory, it becomes a classical trajectory. Thus, in a sense, continuous observation "dequantizes" quantum trajectories. This property is, of course, essentially a consequence of the quantum Zeno effect, which has been shown to inhibit the decay of unstable quantum systems when under continuous observation (see [10, 14, 23, 24]).

The idea lying behind the Bohm approach (Bohm and Hiley [10], Hiley [28], Hiley and collaborators [31, 32], Holland [33]) is the following: let $\Psi = \Psi(\mathbf{r}, t)$ be a wavefunction solution of Schrödinger's equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla_{\mathbf{r}}^2 + V(\mathbf{r}) \right] \Psi.$$

Writing $\Psi$ in polar form $\sqrt{\rho} e^{iS/\hbar}$ Schrödinger's equation is equivalent to the coupled systems of partial differential equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla_{\mathbf{r}} S)^2}{2m} + V(\mathbf{r}) + Q^{\Psi}(\mathbf{r}, t) = 0 \tag{1.1}$$

where

$$Q^{\Psi} = -\frac{\hbar^2}{2m} \frac{\nabla_{\mathbf{r}}^2 \sqrt{|\Psi|}}{\sqrt{|\Psi|}}. \tag{1.2}$$

is Bohm's quantum potential (equation (1.1) is thus mathematically a Hamilton-Jacobi equation), and

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \left( \rho \frac{\nabla_{\mathbf{r}} S}{m} \right) = 0 \tag{1.3}$$

which is an equation of continuity that ensures the conservation of probability. The trajectory of the particle is determined by the equation

$$m\dot{\mathbf{r}}^{\Psi} = \nabla_{\mathbf{r}} S(\mathbf{r}^{\Psi}, t) \quad , \quad \mathbf{r}^{\Psi}(t_0) = \mathbf{r}_0 \tag{1.4}$$

where $\mathbf{r}_0$ is the initial position.

The simple derivation of equations (1.1) to (1.3) obscures a deeper mathematical relation between the Hilbert space formalism of quantum mechanics and the Hamiltonian flows of classical mechanics. This exact relationship has been derived in very general terms by de Gosson and Hiley [21], a paper that generalises the earlier work of de Gosson [17, 18, 20]. Specifically what we show is that there is a one-to-one and onto correspondence between Hamiltonian flows generated by a Hamiltonian $H$ and the strongly continuous unitary one-parameter groups satisfying Schrödinger's equation with Hamiltonian operator $\widetilde{H} = H(x, -i\hbar\nabla_x, t)$ obtained from $H$ by Weyl quantisation. This relation exploits the *metaplectic* representation of the underlying symplectic structure [17, 18, 20]. It is the metaplectic structure that gives rise to the quantum properties. Since the classical and quantum motions are related but different, it was proposed in de Gosson [19] to call the object that obeys the Bohmian law of motion (1.4) a *metatron.*

We choose this term rather than the usual term 'particle', because we are talking about an excitation induced by the metaplectic representation of the underlying Hamiltonian evolution, rather than a classical object. Indeed a deeper investigation suggests that the metatron is more like an invariant feature of an underlying extended process, which elsewhere we have argued that the term *quantum blob* [22] may be more suggestive. However in this paper it is sufficient to regard it as a particle-like object.

We have frequently been asked the question "Didn't Bohm believe that there was an actual classical point-like particle following these quantum trajectories?" The answer is a definite No! For Bohm there was no solid 'particle' either, but instead, at the fundamental level, there was a basic process or activity which left a 'track' in, say, a bubble chamber. Thus the track could be explained by the *enfolding* and *unfolding* of an invariant form in the overall underlying process [7].

Thus rather than seeing the track as the continuous movement of a material particle, it can be regarded as the continuity of a "quasi-local, semi-stable autonomous form" evolving within this unfolding process [28]. This is what we call the *metatron.*

The question we will answer here is the following:

> *What do we see if we perform a continuous observation of the metatron's trajectory?*

Firstly, we will examine the mathematical consequences of continuous observation. We start by assuming the continuously observed trajectory is smooth, and then show that the trajectory we see is the *classical* trajectory determined by the Hamiltonian function

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}). \tag{1.5}$$

Does this mean that Bohmian trajectories in general are not "real ", but must be regarded as "surreal"? *No*, they are not surreal simply because we are making a distinction between what *is*, and what is *observed* by a *physical* measurement.

Two illustrate this point consider the two-slit experiment referred to above. If no attempt is made to observe the trajectories, then they must be like those calculated in Philippidis *et al* [41] to be consistent with the observed interference pattern. However if we observe the motion of the 'particle' as it passes through one of the slits, we will see no wave-like behaviour. Rather we will see a classical trajectory showing no interference effects. In this sense the approach is consistent with standard quantum mechanics. But there is no randomisation of any phase and there is no need for decoherence. Let us now see how this is achieved.

## 2 Bohmian Trajectories Are Hamiltonian

Let us start with the particular case where the metatron is initially localized at a point. In this case the Bohm trajectory is Hamiltonian, a point that we explain in section 2.2 (the general case is slightly more subtle; we refer to the papers by Holland [34, 35] for a thorough discussion of the interpretation of Bohmian trajectories from the Hamiltonian point of view).

We will consider systems of $N$ material particles with the same mass $m$, and work in generalized coordinates $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{p} = (p_1, ..., p_n)$,

$n = 3N$. Suppose that this system is sharply localized at a point $\boldsymbol{x}_0 = (x_{1,0}, ..., x_{n,0})$ at time $t_0$. The classical Hamiltonian function is

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}). \tag{2.1}$$

We have assumed a scalar potential, $V(\boldsymbol{x})$, here for simplicity. However the analysis presented below can be extended to cover the case of a vector potential, and of time-dependent potentials,

The organising field of the system with Hamiltonian (2.1) is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2_{\boldsymbol{x}} + V(\boldsymbol{x})\right]\Psi \quad , \quad \Psi(\boldsymbol{x}, t_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{2.2}$$

where $\nabla_{\boldsymbol{x}}$ is the $n$-dimensional gradient in the variables $x_1, ..., x_n$. The function $\Psi$ is thus just the propagator $G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)$ of the Schrödinger equation. We write $G$ in polar form

$$G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = \sqrt{\rho(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)}e^{\frac{i}{\hbar}S(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)}. \tag{2.3}$$

The equation of motion (1.4) is in this case

$$m\dot{\boldsymbol{x}}^\Psi = \nabla_{\boldsymbol{x}}S(\boldsymbol{x}^\Psi, \boldsymbol{x}_0; t, t_0) \quad , \quad \boldsymbol{x}^\Psi(t_0) = \boldsymbol{x}_0. \tag{2.4}$$

### 2.1 Short-time estimates

We are going to discuss short-time estimates for the functions $S$ and $\rho$. The interest of such estimates is two-fold: they will not only allow us to give a precise statement of the Zeno effect for Bohmian trajectories, but they will also allow us to prove in detail the Hamiltonian character of these trajectories.

We assume that the potential $V$ is at least twice differentiable (with continuous derivatives) in the variables $x_1, ..., x_n$. Writing the quantum propagator in polar form (2.3), the phase $S = S(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)$ is a solution of the quantum Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla_{\boldsymbol{x}}S)^2 + V + Q = 0 \tag{2.5}$$

where the quantum potential $Q$ is given by

$$Q(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = -\frac{\hbar^2}{2m}\frac{\nabla^2_{\boldsymbol{x}}\sqrt{\rho(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)}}{\sqrt{\rho(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)}}. \tag{2.6}$$

(We will see below that $Q$ is indeed defined for $t$ sufficiently close to $t_0$, which is all we need in our discussion of the Hamiltonian character of Bohmian trajectories).

Let $U_{t,t_0}$ be the quantum propagator, that is

$$U_{t,t_0}\psi_0(x) = \int G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)\psi_0(x_0)dx_0. \tag{2.7}$$

Since $U_{t,t_0} = U_{t,t_1}U_{t_1,t_0}$ for all times $t, t_0, t_1$ we have

$$G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = \int G(\boldsymbol{x}, \boldsymbol{x}_1; t, t_1)G(\boldsymbol{x}_1, \boldsymbol{x}_0; t_1, t_0)dx_1. \tag{2.8}$$

Let $N$ be an integer and set $\Delta t = (t - t_0)/N$. It follows from (2.8) by iteration that

$$G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = \int \prod_{j=1}^{N-1} dx_j \prod_{k=0}^{N-1} G(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k; t_k + \Delta t, t_k). \tag{2.9}$$

We emphasize that this formula is *exact*, and involves no approximations. The basic fact is now that for $N \to \infty$ formula (2.9) will still hold if we replace the term $G(\boldsymbol{x}_{k+1}, \boldsymbol{x}_k; t_k + \Delta t, t_k)$ with another function differing from it by a term that goes to zero faster than $\Delta t$ (this can easily be proved using the Lie–Trotter formula; see e.g. Chorin *et al.* [12], de Gosson [19], Appendix B)). It turns out that this condition is satisfied by the short-time propagator

$$G_{\mathrm{sh}}(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = \sqrt{\rho_{\mathrm{sh}}(t - t_0)}e^{\frac{i}{\hbar}W_{\mathrm{sh}}(\boldsymbol{x},\boldsymbol{x}_0;t-t_0)}. \tag{2.10}$$

(see e.g. Schulman [46], Nelson [39], Gaveau *et al.* [15]), Makri and Miller [36, 37], de Gosson [19] (Chapter 7), Schiller [43, 44]). Here

$$\rho_{\mathrm{sh}}(t - t_0) = \left(\frac{m}{2\pi i\hbar(t - t_0)}\right)^n \tag{2.11}$$

and

$$W_{\mathrm{sh}}(t - t_0) = \sum_{j=1}^{n} \frac{m(x_j - x_{0,j})^2}{2(t - t_0)} - \widetilde{V}(\boldsymbol{x}, \boldsymbol{x}_0)(t - t_0) \tag{2.12}$$

where $\widetilde{V}(\boldsymbol{x}, \boldsymbol{x}')$ is the average value of the potential on the line segment $[\boldsymbol{x}', \boldsymbol{x}]$:

$$\widetilde{V}(\boldsymbol{x}, \boldsymbol{x}_0) = \int_0^1 V(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{x}_0)d\lambda. \tag{2.13}$$

In fact one has (see the references above)

$$G_{\mathrm{sh}}(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)=G(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)+O((t-t_0)^2) \tag{2.14}$$

when $t\to t_0\to 0$. In view of the standard inequality $||u|-|z||\leq|u-z|$ valid for all complex numbers $u$ and $z$ it follows that

$$\rho(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)=\rho_{\mathrm{sh}}(t-t_0)+O((t-t_0)^2) \tag{2.15}$$

that is, in view of definition (2.11) of $\rho_{\mathrm{sh}}$,

$$\rho(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)=\left(\frac{m}{2\pi i\hbar(t-t_0)}\right)^n\left(1+O((t-t_0)^{n+2})\right). \tag{2.16}$$

Using the exact series expansions for $S$ given in Makri and Miller [37] one sees that this formula can be differentiated at all orders in the variables $x_j$ and hence, in particular,

$$\nabla^2_{\boldsymbol{x}}\sqrt{\rho(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)}=O((t-t_0)^2) \tag{2.17}$$

from which immediately follows that the quantum potential satisfies the estimate

$$Q(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)=O((t-t_0)^{n/2+2}); \tag{2.18}$$

$Q$ is thus defined for $t$ close to $t_0$, but is *very small*. This estimate has the following important consequence: expanding the phase $S$ in a second order Taylor series around $t=t_0$ we get

$$S(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)=\sum_{j=1}^{n}\frac{m(x_j-x_{0,j})^2}{2(t-t_0)}-\widetilde{V}(\boldsymbol{x},\boldsymbol{x}_0)(t-t_0)+O((t-t_0)^2). \tag{2.19}$$

**Remark 1** *We would like to make the following comment: one often finds in the literature a formula similar to (2.19), but where the average potential is replaced by $V(\frac{1}{2}(x_0+x))$ or $\frac{1}{2}(V(x_0)+V(x))$ (or sometimes bluntly by $V(x)$). We emphasize that all these choices are incorrect and that, when used, lead to approximate propagators which are not correct to order $O(t-t_0)$. See the detailed discussion in Makri and Miller [36]; the necessity for the choice (2.19) was also pointed out by one of us in [19].*

We observe that the quantum potential is absent from formula (2.19); we would actually have obtained the same approximation if we had replaced $S$ with the solution $S_{\mathrm{cl}}$ to the classical Hamilton–Jacobi equation

$$\frac{\partial S_{\mathrm{cl}}}{\partial t}+\frac{1}{2m}(\nabla_{\boldsymbol{x}}S_{\mathrm{cl}})^2+V=0. \tag{2.20}$$

Differentiating twice formula (2.19) with respect to the variables $\boldsymbol{x}_j$ and $\boldsymbol{x}_{0,j}$ the second derivatives of $S$ are given by

$$\frac{\partial^2 S}{\partial x_j \partial x_{0,k}} = \frac{m}{t-t_0}\delta_{jk} + O(t-t_0) \tag{2.21}$$

and hence the Hessian matrix $S_{x,x_0}$ (i.e. the matrix of mixed second derivatives) satisfies

$$\det(S_{\boldsymbol{x},\boldsymbol{x}_0}) = \left(\frac{m}{t-t_0}\right)^n + O(t-t_0). \tag{2.22}$$

Formula (2.19) is the key to the following important asymptotic version of Bohm's equation of motion (2.4):

$$\dot{\boldsymbol{x}}^\Psi = \frac{\boldsymbol{x}^\Psi - \boldsymbol{x}_0}{t-t_0} - \frac{1}{2m}\nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0)(t-t_0) + O((t-t_0)^2)). \tag{2.23}$$

Let us prove this formula. Using the expansion (2.19), formula (2.4) becomes

$$\dot{\boldsymbol{x}}^\Psi = \frac{\boldsymbol{x}^\Psi - \boldsymbol{x}_0}{t-t_0} - \frac{1}{m}\nabla_{\boldsymbol{x}} \tilde{V}(\boldsymbol{x}^\Psi, \boldsymbol{x}_0)(t-t_0) + O((t-t_0)^2). \tag{2.24}$$

Let us show that

$$\nabla_{\boldsymbol{x}} \tilde{V}(\boldsymbol{x}^\Psi, \boldsymbol{x}_0) = \frac{1}{2}\nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0) + O(t-t_0); \tag{2.25}$$

this will complete the proof of formula (2.23). We first note that (2.24) implies in particular that

$$\dot{\boldsymbol{x}}^\Psi = \frac{\boldsymbol{x}^\Psi - \boldsymbol{x}_0}{t-t_0} + O(t-t_0) \tag{2.26}$$

and thus $\boldsymbol{x}^\Psi$ is given by

$$\boldsymbol{x}^\Psi = \boldsymbol{x}_0 + \frac{\boldsymbol{p}_0}{m}(t-t_0) + O((t-t_0)^2) \tag{2.27}$$

where $p_0$ is an arbitrary constant vector. In particular we have $O(\boldsymbol{x}^\Psi - \boldsymbol{x}_0) = O(t-t_0)$ and hence

$$\begin{aligned}\nabla_{\boldsymbol{x}} \tilde{V}(\boldsymbol{x}^\Psi, \boldsymbol{x}_0) &= \nabla_{\boldsymbol{x}} \tilde{V}(\boldsymbol{x}_0, \boldsymbol{x}_0) + O(\boldsymbol{x}^\Psi - \boldsymbol{x}_0) \\ &= \nabla_{\boldsymbol{x}} \tilde{V}(\boldsymbol{x}_0, \boldsymbol{x}_0) + O(t-t_0)\end{aligned} \tag{2.28}$$

from which it follows that

$$\begin{aligned}\nabla_{\boldsymbol{x}}\tilde{V}(\boldsymbol{x}^{\Psi},\boldsymbol{x}_0) &= \int_0^1 \lambda\nabla_{\boldsymbol{x}}V(\lambda\boldsymbol{x}_0+(1-\lambda)\boldsymbol{x}_0)d\lambda + O(t-t_0)\\ &= \frac{1}{2}\nabla_{\boldsymbol{x}}V(\boldsymbol{x}_0)+O(t-t_0) \qquad (2.29)\end{aligned}$$

which is precisely the estimate (2.25).

### 2.2 The Hamiltonian character of Bohmian trajectories

Let $\boldsymbol{p}_0 = (p_{1,0},...,p_{n,0})$ be an arbitrary momentum vector, and set

$$\boldsymbol{p}_0 = -\nabla_{\boldsymbol{x}_0}S(\boldsymbol{x},\boldsymbol{x}_0;t,t_0). \qquad (2.30)$$

In view of formula (2.22), the Hessian of $S$ in the variables $x$ and $x_0$ is invertible for small values of $t$, hence the implicit function theorem implies that (2.30) determines a function $\boldsymbol{x}=\boldsymbol{x}(t)$ (depending on $\boldsymbol{x}_0$ and $t_0$ viewed as parameters), defined by

$$\boldsymbol{p}_0 = -\nabla_{\boldsymbol{x}_0}S(\boldsymbol{x}(t),\boldsymbol{x}_0;t,t_0). \qquad (2.31)$$

Setting

$$\boldsymbol{p}(t) = \nabla_{\boldsymbol{x}}S(\boldsymbol{x}(t),\boldsymbol{x}_0;t,t_0) \qquad (2.32)$$

we claim that the functions $x(t)$ and $p(t)$ thus defined are solutions of the Hamilton equations

$$\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{p}}H^{\Psi}(\boldsymbol{x},\boldsymbol{p},t) \;\;,\;\; \dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{x}}H^{\Psi}(\boldsymbol{x},\boldsymbol{p},t) \qquad (2.33)$$

and that we have $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$, $\boldsymbol{p}(t_0)=\boldsymbol{p}_0$. We are actually going to use classical Hamilton–Jacobi theory (see [2, 16, 19, 20] or any introductory text on analytical mechanics). For notational simplicity we assume that $n=1$. The function $S$ satisfies the equation

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+V(x)-\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}=0; \qquad (2.34)$$

introducing the quantum potential

$$Q^{\Psi} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2} \qquad (2.35)$$

we set $H^\Psi = H + Q^\Psi$ so that (2.34) is just the quantum Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H^\Psi\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \tag{2.36}$$

Differentiating the latter with respect to $p = \partial S/\partial x$ yields, using the chain rule,

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x_0 \partial x} = 0 \tag{2.37}$$

and differentiating the equation (2.31) with respect to time yields

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial^2 S}{\partial x \partial x_0}\dot{x} = 0. \tag{2.38}$$

Subtracting (2.38) from (2.37) we get

$$\frac{\partial^2 S}{\partial x \partial x_0}\left(\frac{\partial H^\Psi}{\partial p} - \dot{x}\right) = 0 \tag{2.39}$$

which produces the first Hamilton equation (2.33) since it is assumed that we have $\partial^2 S/\partial x \partial x_0 \neq 0$. Let us next show that the second Hamilton equation (2.33) is satisfied as well. For this we differentiate the quantum Hamilton–Jacobi equation (2.36) with respect to $x$, which yields

$$\frac{\partial^2 S}{\partial x \partial t} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0. \tag{2.40}$$

Differentiating the equality (2.32) with respect to $t$ we get

$$\frac{\partial^2 S}{\partial t \partial x} = -\dot{p}(t) - \frac{\partial^2 S}{\partial x^2}\dot{x} \tag{2.41}$$

and hence the equation (2.40) can be rewritten

$$-\dot{p}(t) - \frac{\partial^2 S}{\partial x^2}\dot{x} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0. \tag{2.42}$$

Taking into account the relation $\dot{x} = \partial H^\Psi/\partial p$ established above we have

$$-\dot{p}(t) - \frac{\partial H^\Psi}{\partial x} = 0 \tag{2.43}$$

which is precisely the second Hamilton equation (2.33). There remains to show that we have $x(t_0) = x_0$ and $p(t_0) = p_0$. Recall that $x(t)$ is defined by the implicit equation

$$p_0 = -\nabla_{x_0} S(x(t), x_0; t, t_0) \tag{2.44}$$

(equation (2.31)). In view of the short-time estimate (2.19) this means that we have

$$p_0 = \frac{m(x(t) - x_0)}{t - t_0} + O(t - t_0) \tag{2.45}$$

and hence we must have $\lim_{t \to t_0} x(t) = x(t_0) = x_0$. This also implies that $p_0 = m\dot{x}(t_0) = p(t_0)$.

In conclusion we have thus shown that:

> *Bohm's equation of motion (2.4) is equivalent to Hamilton's equations (2.33).*

To complete our discussion, we make two important observations:

- Even when the Hamiltonian function $H$ does not depend explicitly on time, the function $H^\Psi = H + Q^\Psi$ is usually time-dependent (because the quantum potential generally is), so the flow $(f_t^\Psi)$ it determines does not inherit the usual group property $f_t f_{t'} = f_{t+t'}$ of the flow determined by the classical Hamiltonian $H$. One has instead to use the "time-dependent flow" $(f_{t,t'}^\Psi)$, which has a groupoid property in the sense that $f_{t,t'}^\Psi f_{t',t''}^\Psi = f_{t,t''}^\Psi$.

- The time-dependent flow $(f_{t,t'}^\Psi)$ consists of canonical transformations; that is, the Jacobian matrix of $f_{t,t'}^\Psi$ calculated at any point $(x, p)$ where it is defined by a symplectic matrix. This is an immediate consequence of the fact discussed above, namely, that the flow determined by *any* Hamiltonian function has this property.

We have seen that the Bohmian trajectory for a particle initially sharply localized at a point $\boldsymbol{x}_0$ is Hamiltonian, and in fact governed by the Hamilton equations (2.33):

$$\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{p}} H^\Psi(\boldsymbol{x}, \boldsymbol{p}, t) \quad , \quad \dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{x}} H^\Psi(\boldsymbol{x}, \boldsymbol{p}, t). \tag{2.46}$$

The discussion of short-time solutions of Bohm's equation of motion allows us to give approximations to the solution. First, the solutions of the equation $\dot{\boldsymbol{x}} = \nabla_{\boldsymbol{p}} H^\Psi(\boldsymbol{x}, \boldsymbol{p}, t)$ are given by the simple relation

$$\boldsymbol{x}^\Psi(t) = \boldsymbol{x}_0 + \frac{\boldsymbol{p}_0}{m}(t - t_0) + O((t - t_0)^2) \tag{2.47}$$

as was already noticed in (2.27). Then we proved that the momentum $\boldsymbol{p}^\Psi(t) = m\dot{\boldsymbol{x}}^\Psi(t)$ is given by equation (2.23):

$$m\dot{\boldsymbol{x}}^\Psi(t) = \frac{m(\boldsymbol{x}^\Psi(t) - \boldsymbol{x}_0)}{t - t_0} - \frac{1}{2}\nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0)(t - t_0) + O((t - t_0)^2). \tag{2.48}$$

However we cannot solve this equation by inserting the value of $x^{\Psi}(t)$ above since this would lead to an estimate modulo $O(t-t_0)$ not $O((t-t_0)^2)$. What we do is the following: differentiating both sides of the equation (2.48) with respect to $t$ we get

$$\ddot{\boldsymbol{x}}^{\Psi}(t) = \frac{\boldsymbol{x}^{\Psi}(t) - \boldsymbol{x}_0}{(t-t_0)^2} + \frac{\dot{\boldsymbol{x}}^{\Psi}(t)}{t-t_0} - \frac{1}{2m}\nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0) + O(t-t_0) \tag{2.49}$$

that is, replacing $\dot{\boldsymbol{x}}^{\Psi}(t)$ by the value given by (2.48),

$$\dot{\boldsymbol{p}}^{\Psi}(t) = m\ddot{\boldsymbol{x}}^{\Psi}(t) = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0) + O(t-t_0). \tag{2.50}$$

Solving this equation we get

$$\boldsymbol{p}^{\Psi}(t) = \boldsymbol{p}_0 - \nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0)(t-t_0) + O((t-t_0)^2). \tag{2.51}$$

Summarizing, the solutions of the Hamilton equations (2.46) for $H^{\Psi} = H + Q^{\Psi}$ are given by

$$\boldsymbol{x}^{\Psi}(t) = \boldsymbol{x}_0 + \frac{\boldsymbol{p}_0}{m}(t-t_0) + O((t-t_0)^2) \tag{2.52}$$

$$\boldsymbol{p}^{\Psi}(t) = \boldsymbol{p}_0 - \nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0)(t-t_0) + O((t-t_0)^2). \tag{2.53}$$

The observant reader will have noticed that (up to the error term $O((t-t_0)^2)$) there is no trace of the quantum potential $Q^{\Psi}$ in these short-time formulas. Had we replaced the function $H^{\Psi}$ with the classical Hamiltonian $H$ we would actually have obtained exactly the same solutions, up to the $O((t-t_0)^2)$ term.

# 3 Bohmian Zeno Effect

## 3.1 The case of quadratic potentials

Here is an easy case; it is in fact so easy that it is slightly misleading: the Bohmian trajectories are here classical trajectories from the beginning, because the quantum potential vanishes.

Let us assume that the potential $V(\boldsymbol{x})$ is a quadratic form in the position variables, that is

$$V(\boldsymbol{x}) = \frac{1}{2} M\boldsymbol{x} \cdot \boldsymbol{x} \tag{3.1}$$

where $M$ is a symmetric matrix. Using the theory of the metaplectic representation [17, 18, 19, 20] it is well-known that the propagator $G$ is given by the formula

$$G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0) = \left(\tfrac{1}{2\pi i\hbar}\right)^{n/2} i^{m(t,t_0)} \sqrt{|\rho(t,t_0)|} e^{\frac{i}{\hbar} W(\boldsymbol{x},\boldsymbol{x}_0;t,t_0)} \tag{3.2}$$

where $W(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)$ is Hamilton's two-point characteristic function (see e.g. [2, 16]): it is a quadratic form

$$W = \frac{1}{2} P\boldsymbol{x} \cdot \boldsymbol{x} - L\boldsymbol{x} \cdot \boldsymbol{x}_0 + \frac{1}{2} B\boldsymbol{x}_0 \cdot \boldsymbol{x}_0 \tag{3.3}$$

where $P = P(t, t_0)$ and $B = B(t, t_0)$ are symmetric matrices and $L = L(t, t_0)$ is invertible; viewed as function of $x$ it satisfies the Hamilton–Jacobi equation

$$\frac{\partial W}{\partial t} + \frac{(\nabla_{\boldsymbol{x}} W)^2}{2m} + \frac{1}{2} M\boldsymbol{x} \cdot \boldsymbol{x}. \tag{3.4}$$

Moreover, $m(t, t_0)$ is an integer ("Maslov index") and $\rho(t, t_0)$ is the determinant of $L = L(t, t_0)$ (the Van Vleck density). Since $m(t, t_0)$ and $\rho(t, t_0)$ do not depend on $x$, it follows that the quantum potential $Q^\Psi$ determined by the propagator (3.2) is zero. Since we have $H^\Psi = H + Q^\Psi$, we see immediately that the quantum motion is perfectly classical in this case: the quantum equations of motion (2.33) reduce to the ordinary Hamilton equations

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{p}}{m} \ , \ \dot{\boldsymbol{p}} = -M\boldsymbol{x} \tag{3.5}$$

which can be easily integrated: in particular the flow $(f_t)$ they determine is a true flow (because $H = H^\Psi$ is time-independent) and consists of symplectic matrices ([2, 19, 20, 16]). In fact,

$$f_t = e^{tX} \quad , \quad X = \begin{pmatrix} 0_{n\times n} & \frac{1}{m} I_{n\times n} \\ -M & 0_{n\times n} \end{pmatrix}. \tag{3.6}$$

Thus, in the case of quadratic potentials the Bohmian trajectories associated with the propagator are the usual Hamilton trajectories associated with the classical Hamiltonian function of the problem.

Suppose now that we observe "continuously" the time evolution of the metatron –which is so far "quantum"– and try to find out what is recorded by our observation process. Practically this is done by performing repeated position measurements at very short time intervals $\Delta t$. We assume that the recorded trajectory is, in the limit $\Delta t \to 0$, continuous and moreover smooth; by this we mean that we can assign at every point a velocity vector (we are thus excluding Brownian motion-type behavior). Let us choose a time interval $[0, t]$ (typically $t = 1\,\mathrm{s}$) and subdivide it in a sequence of $N$ intervals

$$[0, \Delta t], \ [\Delta t, 2\Delta t], \ [2\Delta t, 3\Delta t], \cdots, \ [(N-1)\Delta t, N\Delta t] \tag{3.7}$$

with $\Delta t = t/N$; the integer $N$ is assumed to be very large (for instance $N \simeq 10^6 - 10^8$). Assume that a measurement at time $t_0 = 0$ localizes the particle at a point $\boldsymbol{x}_0$ it will be detected at a point $\boldsymbol{x}_1$ after time $\Delta t$; its momentum is $\boldsymbol{p}_1$ and we have $(\boldsymbol{x}_1, \boldsymbol{p}_1) = f_{\Delta t}(\boldsymbol{x}_0, \boldsymbol{p}_0)$. We now repeat the procedure, replacing $\boldsymbol{x}_0$ by $\boldsymbol{x}_1$; since the observed trajectory is assumed to be smooth the initial momentum will be $\boldsymbol{p}_1$ and after time $\Delta t$ a new measurement is performed, and we find the particle at $\boldsymbol{x}_2$ with momentum $\boldsymbol{p}_2$ such that $(\boldsymbol{x}_2, \boldsymbol{p}_2) = f_{\Delta t}(\boldsymbol{x}_1, \boldsymbol{p}_1) = f_{\Delta t} f_{\Delta t}(\boldsymbol{x}_0, \boldsymbol{p}_0)$. Repeating the same process until time $t = N\Delta t$ we find a series of points in space which the particle takes as positions one after another[1] that $(\boldsymbol{x}_N, \boldsymbol{p}_N) = (f_{\Delta t})^N(\boldsymbol{x}_0, \boldsymbol{p}_0)$. But in view of the group property $f_t f_{t'} = f_{t+t'}$ of the flow we have $(f_{\Delta t})^N = f_{N\Delta t} = f_t$ and hence $(\boldsymbol{x}_N, \boldsymbol{p}_N) = f_t(\boldsymbol{x}_0, \boldsymbol{p}_0)$. The observed Bohmian trajectory is thus the classical trajectory predicted by Hamilton's equations.

## 3.2 The general case

In generalizing the discussion above to arbitrary potentials, $V(\boldsymbol{x})$, there are two difficulties. The first is that we do not have exact equations for the Bohmian trajectory, but only short-time approximations. The second is that the Hamilton equations for $\boldsymbol{x}^\Psi$ and $\boldsymbol{p}^\Psi$ no longer determine a flow having a group property because the Hamiltonian $H^\Psi$ is time-dependent. Nevertheless the material we have developed so far is actually sufficient to show that the observed trajectory is the classical one.

The key will be the theory of Lie–Trotter algorithms which is a powerful method for constructing exact solutions from short-time estimates. The method goes back to early work of Trotter [47] elaborating on Sophus Lie's proof of the exponential matrix formula $e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N$; see Chorin et al. [12] for a detailed and rigorous study; we have summarized the main ideas in the Appendix B of [19]); also see Nelson [40]. (We mention that there exists an operator variant of this procedure, called the Trotter–Kato formula).

Let us begin by introducing some notation. We have seen that the datum of the propagator $G_0 = G(\boldsymbol{x}, \boldsymbol{x}_0; t, t_0)$ determines a quantum potential $Q^\Psi$ and thus Hamilton equations (2.33) associated with $H^\Psi = H + Q^\Psi$. We now choose $t_0 = 0$ and denote the corresponding quantum potential by $Q^0$ and set $H^0 = H + Q^0$. After time $\Delta t$ we make a position measurement and find that the particle is located at $\boldsymbol{x}_1$. The future evolution of the particle is now

[1] In conformity with W. Heisenberg's statement: "By path we understand a series of points in space which the electron takes as 'positions' one after another" [25]

governed by the new propagator $G_1 = G(\boldsymbol{x}, \boldsymbol{x}_1; t, t_0)$, leading to a new quantum potential $Q^1$ and to a new Hamiltonian $H^1$; repeating this until time $t$ we thus have a sequence of points $\boldsymbol{x}_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_N = \boldsymbol{x}$ and a corresponding sequence of Hamiltonian functions $H^0, H^1, ..., H^N$ determined by the quantum potentials $Q^0, Q^1, ..., Q^N$. We denote by $(f^0_{t,t_0})$, $(f^1_{t,t_1})$,...,$(f^{N-1}_{t,t_{N-1}})$ the time dependent flows determined by the Hamiltonian functions $H^0, H^1, ..., H^N$; we have set here $t_1 = t_0 + \Delta t$, $t_2 = t_1 + \Delta t$ and so on.

Repeating the observation procedure explained in the case of quadratic potentials, we get in this case a sequence of successive equalities

$$\begin{aligned}(\boldsymbol{x}_1, \boldsymbol{p}_1) &= f^0_{t_1,t_0}(\boldsymbol{x}_0, \boldsymbol{p}_0)\\ (\boldsymbol{x}_2, \boldsymbol{p}_2) &= f^1_{t_2,t_1}(\boldsymbol{x}_1, \boldsymbol{p}_1)\\ &\cdots\cdots\cdots\cdots\\ (\boldsymbol{x}, \boldsymbol{p}) &= f^{N-1}_{t,t_{N-1}}(\boldsymbol{x}_{N-1}, \boldsymbol{p}_{N-1})\end{aligned}$$

which implies that the final point $\boldsymbol{x} = \boldsymbol{x}_N$ observed at time $t$ is expressed in terms of the initial point $\boldsymbol{x}_0$ by the formula

$$(\boldsymbol{x}, \boldsymbol{p}) = f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,t_0}(\boldsymbol{x}_0, \boldsymbol{p}_0). \tag{3.8}$$

Denote now by $(g^0_{t,t_0})$, $(g^1_{t,t_1})$,...,$(g^{N-1}_{t,t_{N-1}})$ the approximate flows determined by the equations

$$\begin{aligned}(\boldsymbol{x}_1, \boldsymbol{p}_1) &= (\boldsymbol{x}_0 + \frac{\boldsymbol{p}_0}{m}\Delta t, \boldsymbol{p}_0 - \nabla_{\boldsymbol{x}} V(\boldsymbol{x}_0)\Delta t)\\ (\boldsymbol{x}_2, \boldsymbol{p}_2) &= (\boldsymbol{x}_1 + \frac{\boldsymbol{p}_1}{m}\Delta t, \boldsymbol{p}_1 - \nabla_{x} V(\boldsymbol{x}_1)\Delta t)\\ &\cdots\cdots\cdots\cdots\\ (\boldsymbol{x}, \boldsymbol{p}) &= (\boldsymbol{x}_{N-1} + \frac{\boldsymbol{p}_{N-1}}{m}\Delta t, \boldsymbol{p}_{N-1} - \nabla_{\boldsymbol{x}} V(\boldsymbol{x}_{N-1})\Delta t).\end{aligned}$$

Invoking the Lie–Trotter formula, the sequence of estimates

$$f^0_{t_k,t_{k-1}}(\boldsymbol{x}_{k-1}, \boldsymbol{p}_{k-1}) - g^0_{t_k,t_{k-1}}(\boldsymbol{x}_{k-1}, \boldsymbol{p}_{k-1}) = O(\Delta t^2) \tag{3.9}$$

implies that we have

$$\lim_{N\to\infty} g^{N-1}_{t,t_{N-1}} \cdots g^1_{t_2,t_1} g^0_{t_1,0}(\boldsymbol{x}_0, \boldsymbol{p}_0) = \lim_{N\to\infty} f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,0}(\boldsymbol{x}_0, \boldsymbol{p}_0) \tag{3.10}$$

The argument goes as follows (for a detailed proof see [19]): since we have $g^k_{t_k,t_{k-1}} = f^k_{t_k,t_{k-1}} + O(\Delta t^2)$ the product is approximated by

$$g^{N-1}_{t,t_{N-1}} \cdots g^1_{t_2,t_1} g^0_{t_1,t_0} = f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,t_0} + NO(\Delta t^2) \tag{3.11}$$

and since $\Delta t = t/N$ we have $NO(\Delta t^2) = O(\Delta t)$ which goes to zero when $N \to \infty$.

Now, recall our remark that the quantum potential is absent from the approximate flows $g^k_{t_k,t_{k-1}}$; using again the Lie–Trotter formula together with short-time approximations to the Hamiltonian flow $(f_t)$ determined by the classical Hamiltonian $H$, we get

$$\lim_{N\to\infty} g^{N-1}_{t,t_{N-1}} \cdots g^1_{t_2,t_1} g^0_{t_1,0}(\boldsymbol{x}_0, \boldsymbol{p}_0) = f_t \tag{3.12}$$

and hence

$$\lim_{N\to\infty} f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,0}(\boldsymbol{x}_0, \boldsymbol{p}_0) = f_t \tag{3.13}$$

which shows that the observed trajectory is the classical one.

# 4 Conclusion.

In this paper we have shown how a detailed mathematical examination of the deeper symplectic structure that underlies the Bohm approach predicts that if a quantum particle is watched continuously, it will follow a classical trajectory.

The intuitive idea lying behind this result becomes clear once one realises that it is the appearance of the quantum potential energy that distinguishes quantum behaviour from classical behaviour. Indeed this is very obvious if we examine equation (1.1) and compare it with the classical Hamilton-Jacobi equation. The essential difference is the appearance of the term, $Q^\Psi$, in equation (1.1). This means that when $Q^\Psi$ is negligible compared with the kinetic energy, the equation simply reduces to the classical Hamilton-Jacobi equation. Further discussion of the relationship between the quantum and classical domains in the Bohm approach can be found in Bohm and Hiley [9].

Hiley and Aziz Mufti [27] give an interesting example of one way the quantum potential can become negligible over time. In a simplified cosmological model, they show how, in an inflationary scenario, the quantum potential becomes negligible so that any quantum behaviour becomes classical in later stages of the inflation. In this paper we have shown another way in which the quantum potential can be suppressed, namely, by the continuous observation.

To see this, we must examine equations (2.52) and (2.53), which are exact to $O\left((\Delta t)^2\right)$. Notice that there is no quantum potential present in either equation. Only when we allow higher order terms does the quantum potential appear. Thus if we make a succession of position observations in a short enough time without deflecting the particle significantly, then no quantum potential will appear and the trajectory will be a classical trajectory. In other words the quantum Zeno effect arises because $Q^{\Psi}$ is prevented from contributing to the process.

Another illustration of how continuous observations gives rise to a quantum Zeno effect, which, in this case, suppresses an atomic transition, has already been given in Bohm and Hiley [10]. They considered the transition in an Auger-like particle and showed that the perturbed wave function, which is proportional to $\Delta t$ for times less that $1/\Delta E$, ($\Delta E$ is the energy released in the transition) will never become large and therefore cannot make a significant contribution to the quantum potential. Again for this reason no transition will take place.

Finally we must consider how we can physically produce such a continuous observation. Luckily this question has already been discussed as far back as 1929 when Mott [38] asked how the Schrödinger equation can account for the $\alpha$-particle tracks seen in a Wilson cloud chamber. This problem has also been discussed by Heisenberg [26] and, in a more modern setting, by Bell [3].

Mott's considered the $\alpha$-particle being produced by a decaying nucleus. The initial wave function of the $\alpha$-particle is spherically symmetric and the problem is to show how, using the wave function alone, we can account for the classical particle-like trajectory produced in the cloud chamber. To do this we first assume that the ionisation of the gas atoms act as a 'measuring device', demanded by standard quantum mechanics.

The task is to show that the gas atoms cannot be ionised unless they lie in a straight line emanating from the radioactive nucleus. The actual analysis presented by Mott [38], the details of which we will not discuss here, requires solutions of the Schrödinger equation using the Born approximation. Also to simplify the treatment, the interactions between the $\alpha$-particle and the nuclei of the atoms are omitted, which means that any deviations from the straight line are neglected. This is exactly the case we have considered above.

In our approach we do not need to have a specific measuring process. Indeed one of the merits of the Bohm approach is that there is no need to treat a 'measurement' as different from any other quantum interaction. A 'measurement' is simply a special case of a quantum transition designed to make manifest a chosen property of the system. This point has been discussed fully in Bohm and Hiley [8, 10].

Thus, in the case considered here, it is sufficient to have any process that interrupts the flow sufficiently rapidly, but allows the flow to continue smoothly for a very short time between interruptions. If the frequency between interruptions were longer, terms greater than $O\left((t-t_0)^2\right)$ would appear, giving rise to a quantum potential that could change the overall flow. In this case we would need to introduce an interaction Hamiltonian to describe the nature of the change of flow. However the work of Mott [38] shows that the Wilson cloud chamber operating at normal gas pressures can be regarded as providing the type of continuous measurement we require.

This discussion shows that the Bohm model has a very different way of arriving at the classical limit than the prevailing view based on decoherence. In our view the main difficulty in using decoherence is that it merely destroys the off-diagonal elements of the density matrix but it does not explain how the classical equations of motion arise. It continues to describe classical objects using wave functions, a criticism that has already been made by Primas [42].

The method we present in this paper is a further example of how the relation between the symplectic and metaplectic representations discussed in our earlier paper [21] holds a further clue of the relationship between the quantum and classical domains. It is when the global properties of the covering (metaplectic) group become unimportant that the classical world emerges. As has been pointed out by Hiley [29, 30], the Bohm approach has a close relationship to the Moyal approach. This supplements the work of de Gosson [20] who show exactly how the Wigner-Moyal transformation is related to the mathematical structure we are exploiting here. The Moyal approach involves a deformed Poisson algebra from which the classical limit emerges in a very simple way, namely, in those situations where the deformation parameter can be considered to be small which is essentially similar to neglecting the quantum potential.

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